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Bi-Hamiltonian systems of deformation type

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Abstract

In this paper, after some recalls about Poisson cohomology, we first study what the general method is in order to obtain a bi-Hamiltonian formulation of a given Hamiltonian system by means of a deformation. Then we show that the bi-Hamiltonian formulation which results from the deformation of a Poisson structure by means of a suitable non-Noether symmetry cannot explain the complete integrability for a large class of Arnold–Liouville integrable systems; next we prove that the deformation must be made in this context by a suitable mastersymmetry. At last, we give several examples.

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1. Introduction

One can have two approaches concerning relations between bi-Hamiltonian structures and completely integrable Hamiltonian systems. Firstly, we can have a practical one and

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try to prove the complete integrability of a given Hamiltonian system by means of the discovery of a suitable bi-Hamiltonian structure. Secondly, we can have a more theoretical approach by explaining the complete integrability of a well-known completely integrable Hamiltonian system: one show that a known involutive family of constants of motion (which gives the integrability) comes from a recursion operator associated with a bi-Hamiltonian structure. The former should be very interesting but unfortunately it seems utopian. Indeed, without the knowledge of an involutive family of constants of motion, it appears impossible (or extremely difficult, at least) to find such a bi-Hamiltonian structure and, with such knowledge, we are in the situation of the latter point of view! Therefore, in this paper we are only concerned with this second problem.

So, we consider a completely integrable Hamiltonian system (M, Π, H), where M is a smooth manifold, Π a Poisson tensor and H a smooth function on M, with an involutive family of functionally independent constants of motion $(f_1, \ldots, f_n)$. We examine the existence of a bi-Hamiltonian structure for it, i.e. the existence of a second Poisson structure Π', compatible with the initial Poisson tensor Π, and so that the Hamiltonian vector field X_{H} for Π is also Hamiltonian for Π' (condition which is locally equivalent to the fact that Π' is X_{H}-invariant, i.e. X_{H} is a Poisson automorphism). One of us has already studied such a problem in the context of Arnold–Liouville systems i.e. when the fibers of the application

$$F : M \longrightarrow \mathbb{R}^n, \quad x \mapsto (f_1(x), \ldots, f_n(x))$$

are compact sets [1,2], and we know that, in this situation, the existence of such a bi-Hamiltonian structure is very rare. The compacity of fibers is very constraining and, in the absence of such a condition, the situation is less rigid. In particular, for non-singular Hamiltonian, the existence is always locally satisfied.

In this paper we are concerned with the case where a bi-Hamiltonian formulation for a given completely integrable Hamiltonian system is obtained by a deformation of the initial Poisson structure. In fact, it is well known from Magri's works in the 1980s that the infinitesimal deformation $\Pi' = [Z, \Pi]$ of a Poisson structure Π always verifies $[\Pi, \Pi'] = 0$ where $[,]$ is the Schouten–Nijenhuis bracket. So, in the case where Π' is also Poisson, it provides a pair of compatible Poisson tensors. Locally, this condition is not constraining because every compatible Poisson structure is obtained by a deformation of the initial one. In the situation of an Arnold–Liouville integrable system (fibers are compact) we are in the model of $M = U \times \mathbb{T}^n$, where U is an open ball of $\mathbb{R}^n$, $\mathbb{T}^n$ the n-dimensional torus and Π the canonical structure $\sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial \theta_i}$ where $(q_1, \ldots, q_n, \theta_1, \ldots, \theta_n)$ are action-angle coordinates on M; in this case, if Π' is a Poisson tensor, compatible with Π, Π' is not necessarily obtained by a deformation however there are some real constants c_{ij} and a field Z, so that

$$\Pi' = [Z, \Pi] + \sum_{i<j} c_{ij} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial q_j}.$$

Thus, even in this last situation, we are not far from a deformation.

In a recent paper [3] in this journal, Chavchanidze studied the notion of non-Noether symmetry and its link with bi-Hamiltonian systems. Such a symmetry for a Hamiltonian system (M, Π, H) is (from an infinitesimal point of view) a field Z verifying $[Z, X_{\mathrm{H}}] = 0$ (i.e. a symmetry of the Hamiltonian vector field X_{H}) and $[Z, \Pi] \neq 0$ (non-Noether). Under the condition $[[Z, [Z, \Pi]], \Pi] = 0$, Chavchanidze shows that the second bivector defines a Poisson structure on M and – because of the condition of symmetry, this second Poisson structure is X_{H}-invariant – leading to a non-trivial bi-Hamiltonian system with all its well-known classical properties, non-trivial because symmetry is non-Noether. Chavchanidze's paper is at the origin of the present work.

More precisely, in Section 2, we briefly recall some basic facts on Poisson cohomology because it is the good framework to study deformations of Poisson structures.

In Section 3, we make a general study of bi-Hamiltonian systems which are obtained by a deformation: we characterize the suitable fields and, joined to a cohomological interpretation, it leads to a synthetic presentation of bi-Hamiltonian systems. With this approach, these systems appear as a moduli space of solutions of partial differential equations. A new cohomology, associated with the Hamiltonian function is introduced, leading to a bidifferential calculus with the classical Poisson cohomology.

In Section 4, we study the case of completely integrable Hamiltonian systems with compact fibers (Arnold–Liouville context) and show on the one hand that deformations using non-Noether symmetries are not relevant in this situation for a large class of systems. As the spectrum of the recursion operator is necessary constant it cannot allow us to find an interesting family of constants of motion. On the other hand we show that the suitable fields for interesting deformations must be mastersymmetries.

In the last section, we give several examples to illustrate our purpose with systems defined on $\mathbb{R}^{2n}$ with canonical symplectic form and for systems defined on semi-simple Lie algebra of compact type endowed with the Kostant–Kirillov–Souriau Poisson structure; more accurately we deal with the Toda and relativistic Toda lattices and the Euler equation.

2. Poisson compatibility and Poisson cohomology

2.1. Compatible Poisson structures and deformations

Let (M, Π) a *Poisson manifold*. For $k \in \mathbb{N}$, we note $\mathfrak{X}^k(M)$ the space of smooth k-vector fields on M; in particular $\mathfrak{X}^0(M) = \mathcal{C}^\infty(M)$ is the space of smooth functions on M and $\mathfrak{X}^1(M) = \mathfrak{X}(M)$ the space of smooth vector fields on M. Let

$$\mathfrak{X}^*(M) = \bigcup_{k\in\mathbb{N}} \mathfrak{X}^k(M).$$

We can define on $\mathfrak{X}^*(M)$ a bracket $[,]$, called the *Schouten–Nijenhuis bracket*, graded extension of the Lie bracket on $\mathfrak{X}(M)$. Recall that, using the Koszul sign convention [7], this Schouten–Nijenhuis bracket satisfies the following identities for all P, Q, R, respectively, in $\mathfrak{X}^p(M)$, $\mathfrak{X}^q(M)$, $\mathfrak{X}^r(M)$:

$$[P, Q] = -(-1)^{(p-1)(q-1)}[Q, P], \qquad (1)$$

$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(p-1)q} Q \wedge [P, R], \tag{2}$$

$$(-1)^{(p-1)(r-1)}[P, [Q, R]] + (-1)^{(q-1)(p-1)}[Q, [R, P]] + (-1)^{(r-1)(q-1)}[R, [P, Q]] = 0. \tag{3}$$

The last one is the so-called *graded Jacobi identity*. In terms of this bracket, the condition for Π to be Poisson is written $[\Pi, \Pi] = 0$.

Because of the *graded Jacobi identity* on $\mathfrak{X}^*(M)$, if Z is a vector field on M, we have

$$0 = [Z, [\Pi, \Pi]] = 2[[Z, \Pi], \Pi]$$

and so the 2-vector field Π' defined by $\Pi' = [Z, \Pi]$ verifies $[\Pi, \Pi'] = 0$.

Recall that two Poisson tensors Π_0 and Π_1 on M are said *compatible* if every linear combination $\lambda_0\Pi_0 + \lambda_1\Pi_1$ is also Poisson; this fact is clearly equivalent to the condition $[\Pi_0, \Pi_1] = 0$. For this reason, we will say that a p-vector field P and a q-vector field Q are *compatible* if $[P, Q] = 0$. Then, the previous remark about $\Pi' = [Z, \Pi]$ leads to the conclusion that the *infinitesimal deformation* Π' of the Poisson tensor Π is always compatible with Π. In particular, if Π' is a Poisson tensor, we obtain on M two compatible Poisson tensors. So, we define the following definition.

Definition 2.1. If Π is a Poisson tensor and Z a vector field so that $\Pi' = [Z, \Pi]$ is also Poisson, then Π' is said of deformation type and (Π, Π') is called a compatible Poisson pair of deformation type.

However, in general, $\Pi' = [Z, \Pi]$ is not Poisson and we will look at this problem in Section 3.

2.2. Poisson cohomology

Let (M, Π) a Poisson manifold. Consider for $k \in \mathbb{N}$, the application

$$d_\Pi^k : \mathfrak{X}^k(M) \longrightarrow \mathfrak{X}^{k+1}(M), \qquad \Lambda \mapsto [\Pi, \Lambda].$$

We have for all integer k, the relation $d_\Pi^{k+1} \circ d_\Pi^k = 0$ and so, we can define the following cochains complex:

$$\mathfrak{X}^0(M) \overset{d_\Pi^0}{\rightarrow} \mathfrak{X}^1(M) \overset{d_\Pi^1}{\rightarrow} \mathfrak{X}^2(M) \overset{d_\Pi^2}{\rightarrow} \mathfrak{X}^3(M) \longrightarrow \cdots \tag{4}$$

and the cohomology spaces

$$H_\Pi^k(M) = \operatorname{Ker} d_\Pi^k / \operatorname{Im} d_\Pi^{k-1} \tag{5}$$

with the convention $\operatorname{Im} d_\Pi^{-1} = \{0\}$. This cohomology $H_\Pi^*(M)$ is called the *Poisson cohomology* and was introduced by Lichnerowicz.

Among these cohomological spaces, the most interesting for our study is the second one $H_\Pi^2(M)$, because, if it is trivial, the only 2-tensors compatible with Π are obtained by infinitesimal deformations of Π, in particular, all compatible Poisson structures are of deformation type.

In general, the calculation of the Poisson cohomology is very difficult; for some results in this area, one can report to [10,11] and [15]. Nevertheless, if Π is non-degenerate (and so M is symplectic), the Poisson cohomology of M is isomorphic to the De-Rham one. For a closed (i.e. compact boundaryless) symplectic manifold, it is impossible to have $H^2_{\mathrm{DR}}(M)=\{0\}$ and so $H^2_{\Pi}(M)\neq\{0\}$; however, it is true for all local study because we can suppose that we are on some open ball of $\mathbb{R}^{2n}$ endowed with the canonical symplectic form $\omega=\sum \mathrm{d}q_i\wedge \mathrm{d}p_i$. Thus, in this case, all Poisson structures compatible with the canonical one Π_ω, are necessarily obtained by an infinitesimal deformation of Π_ω.

Another situation where the second Poisson cohomological space is trivial is provided by semi-simple Lie algebras of compact type. Indeed, if $\mathcal{G}$ is the Lie algebra of a compact Lie group and Π the Lie–Poisson structure of $\mathcal{G}^*$, also known as the *Kostant–Kirillov–Souriau* Poisson structure and defined for $f, g\in\mathcal{C}^\infty(\mathcal{G}^*)$ and $\alpha\in\mathcal{G}^*$ by the bracket

$$\{f,g\}_\Pi(\alpha)=\langle\alpha,[\mathrm{d}f(\alpha),\mathrm{d}g(\alpha)]\rangle, \tag{6}$$

we have for all integer k (see [15, p. 69]),

$$H^k_\Pi(\mathcal{G}^*)=H^k(\mathcal{G})\otimes\{\text{Casimir functions of}\,(\mathcal{G}^*,\Pi)\},$$

where $H^k(\mathcal{G})$ is the cohomology of the Lie algebra $\mathcal{G}$. In particular, by classical results on this last cohomology, if $\mathcal{G}$ is a semi-simple Lie algebra of compact type, we have $H^2(\mathcal{G})=\{0\}$ and so $H^2_\Pi(\mathcal{G}^*)=\{0\}$.

3. Bi-Hamiltonian systems and Poisson–Hamilton cohomology

3.1. The Yang–Baxter–Poisson equation

A deformed tensor $\Pi'=[Z,\Pi]$ is Poisson if, and only if, $[\Pi',\Pi']=0$, i.e. $[[Z,\Pi],\Pi']=0$; this condition is equivalent – by the graded Jacobi identity and the compatibility of Π and Π' – to the equation

$$[[Z,[Z,\Pi]],\Pi]=0. \tag{7}$$

So we can state the following proposition.

Proposition 3.1. *Let* (M,Π) *a Poisson manifold and* Z *a vector field satisfying the condition* $[[Z,[Z,\Pi]],\Pi]=0$. *Then* $(\Pi,\Pi':=[Z,\Pi])$ *defines a compatible Poisson pair of deformation type on* M.

Remark 3.2.

(a) As we already said in Section 1, the relation (7) is the condition assumed by Chavchanidze [3] on non-Noether symmetries to get his results.

(b) In his paper (cf. loc. cit), Chavchanidze compared Eq. (7) to the classical Yang–Baxter Equation. For this reason, we suggest to call *Yang–Baxter–Poisson fields*, solutions of Eq. (7).

(c) Eq. (7) is in particular satisfied in the special case where

$$[Z,[Z,\Pi]]=0. \tag{8}$$

Such a field will be called a *special Yang–Baxter–Poisson field*.

The previous remarks lead to try to find Yang–Baxter–Poisson fields or special Yang–Baxter–Poisson fields i.e. solutions of Eqs. (7) and (8), that we shall, respectively, name the Yang–Baxter–Poisson Equation (shortly referred as (YBPE)) and the special Yang–Baxter–Poisson equation (shortly referred as (SYBPE)), namely

$$[[Z,[Z,\Pi]],\Pi]=0 \quad \text{(YBPE)}$$

and

$$[Z,[Z,\Pi]]=0 \quad \text{(SYBPE)}.$$

Let us begin with a remark; on a surface (i.e. a 2-manifold) all 2-vector fields are Poisson. But, if M is a $2n$-dimensional symplectic manifold, by Darboux theorem, M is locally a product of n symplectic surfaces and so all fields on M which are adapted to this local decomposition are solutions of (YBPE). In other words, if $(q_1,p_1,q_2,p_2,\ldots,q_n,p_n)$ are local Darboux coordinates on an open set U, fields of type

$$Z=\sum_{i=1}^{n} f_i(q_i,p_i)\frac{\partial}{\partial q_i}+g_i(q_i,p_i)\frac{\partial}{\partial p_i}$$

are solutions of (YBPE) on U. For such a field Z, the compatible deformed Poisson structure $\Pi'=[Z,\Pi]$ obtained on U is written as

$$\begin{aligned}\Pi'=[Z,\Pi]&=\left[\sum_{i=1}^{n} f_i(q_i,p_i)\frac{\partial}{\partial q_i}+g_i(q_i,p_i)\frac{\partial}{\partial p_i},\sum_{j=1}^{n}\frac{\partial}{\partial q_j}\wedge\frac{\partial}{\partial p_j}\right]\\&=\sum_{i,j=1}^{n}\left[f_i\frac{\partial}{\partial q_i}+g_i\frac{\partial}{\partial p_i},\frac{\partial}{\partial q_j}\right]\wedge\frac{\partial}{\partial p_j}+\frac{\partial}{\partial q_j}\wedge\left[f_i\frac{\partial}{\partial q_i}+g_i\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}\right]\\&=-\sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial q_i}+\frac{\partial g_i}{\partial p_i}\right)\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i}.\end{aligned} \tag{9}$$

Another remark is the following one: let us search solutions of (SYBPE) $Z=fY$ where Y is an infinitesimal automorphism of Π. In this case, because $[Y,\Pi]=0$, we have—if we note X_f the Hamiltonian vector field $[\Pi,f]$ associated with Hamiltonian function f

$$\begin{aligned}0 &= [fY,[fY,\Pi]] = -[fY,[\Pi,f]\wedge Y] = -f[Y,X_f]\wedge Y + (Y\cdot f)X_f\wedge Y\\ &= (-fX_{Y\cdot f} + (Y\cdot f)X_f)\wedge Y.\end{aligned} \tag{10}$$

Moreover, the deformed Poisson structure is given by

$$\Pi' = -X_f \wedge Y \tag{11}$$

and so has a rank 0 or 2. Two particular cases appear when we look at Eq. (10), namely $Y\cdot f = 0$ and $Y\cdot f = f$. Indeed, in this two cases we obtain solutions of (SYBPE). For example, to find Poisson structures compatible with the canonical one $\Pi = \sum_{i=1}^{n} \frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i}$ on $\mathbb{R}^{2n}$, it suffices to choose a Hamiltonian vector field Y, for instance $Y = X_{q_1} = \frac{\partial}{\partial p_1}$, a function f which verifies $Y\cdot f = 0$ or $Y\cdot f = f$, i.e. with the given example, $f = f(q_1, q_2, p_2, \dots, q_n, p_n)$ in the first case and $f = \mathrm{e}^{p_1} g(q_1, q_2, p_2, \dots, q_n, p_n)$ in the second one, and to deform Π by means of the field $Z = fY$ according to the formula (11). Now, if we have n independent automorphisms $Y_1, \dots, Y_n$ of Π and n functionally independent first integrals $f_1, \dots, f_n$ for the fields $Y_1, \dots, Y_n$, with $Y_1, \dots, Y_n, X_{f_1}, \dots, X_{f_n}$ independent, then the deformed structure $\Pi' = [Z,\Pi]$ where $Z = f_1Y_1 + \cdots + f_nY_n$, is a non-degenerate Poisson structure, compatible with Π. For example, we are in such a situation when we have a completely integrable Hamiltonian system: by Jacobi–Lie–Caratheodory theorem [9], if $(f_1, \dots, f_n)$ is an involutive family, there are locally, functions $g_1, \dots, g_n$ such that $f_1, \dots, f_n, g_1, \dots, g_n$ are canonical coordinates and so we can take $Y_1 = X_{g_2}, \dots, Y_{n-1} = X_{g_n}, Y_n = X_{g_1}$.

3.2. The Yang–Baxter–Poisson moduli space

Recall first the classical definition of a *bi-Hamiltonian system*.

Definition 3.3. A Hamiltonian system (M,Π,H) is said to admit a bi-Hamiltonian formulation if there is a bivector Π' satisfying the three following conditions:

(a) $[\Pi',\Pi'] = 0$ (Poisson condition)
(b) $[\Pi,\Pi'] = 0$ (compatibility condition)
(c) $[[H,\Pi],\Pi'] = 0$ (bi-Hamiltonian condition)

In this case we will call (M,Π,Π',H) a bi-Hamiltonian system.

Remark 3.4. The condition (c) in the previous definition says that the Hamiltonian vector field $X_{\mathrm{H}} := [H,\Pi]$ is an infinitesimal automorphism of Π'; among these automorphisms, we find Hamiltonian vector fields relatively to Π', and locally all automorphisms are such Hamiltonian fields, i.e. $[H,\Pi] = [H',\Pi']$ for some function H'. This fact justifies the word “bi-Hamiltonian” used in this definition.

We can also remark that using the graded Jacobi identity for Schouten–Nijenhuis bracket and the condition (b), the condition (c) gives immediately that the Hamiltonian vector field $[H,\Pi']$ is an infinitesimal automorphism of Π.

Definition 3.5. A bi-Hamiltonian formulation for the system (M, Π, H) will be called of deformation type if Π' is obtained by deformation of Π i.e. $\Pi' = [Z, \Pi]$ for a suitable field Z.

Proposition 3.6. *A Hamiltonian system* (M, Π, H) *admits a bi-Hamiltonian formulation of deformation type if, and only if, there is a vector field* $Z \in \mathfrak{X}(M)$ *satisfying the two following conditions*:

(i) $[[Z, [Z, \Pi]], \Pi] = 0$ (*YBPE*)
(ii) $[[Z, [H, \Pi]], \Pi] = 0$, *i.e.* $[Z, X_{\mathrm{H}}]$ *is an infinitesimal automorphism of* Π. *We refer to this condition as* (*PHE*) (*for Poisson–Hamilton equation*).

In this case the second Poisson structure Π' *of Definition* 3.3 *is given by* $\Pi' = [Z, \Pi]$.

Proof. We have already seen that the condition (i), i.e. the (YBPE) is a necessary and sufficient condition to have $\Pi' := [Z, \Pi]$ Poisson. The condition of compatibility is in the case of a deformation automatically satisfied. Now, for $\Pi' := [Z, \Pi]$, the condition (c) of Definition 3.3 is written $[X_{\mathrm{H}}, [Z, \Pi]] = 0$ and so using the graded Jacobi identity for the Schouten–Nijenhuis bracket and the fact that X_{H} is an infinitesimal automorphism of Π, we see immediately that it is equivalent to the condition (ii). □

Remark 3.7. In [3], Chavchanidze defines a *non-Noether symmetry* as a field Z so that $[Z, X_{\mathrm{H}}] = 0$ and $[Z, \Pi] \neq 0$. Such a field is a trivial solution of (PHE). We will reconsider largely this concept in Section 4.

It is so natural to define, for a Hamiltonian system (M, Π, H), the following set:

$$\mathcal{S} = \{Z \in \mathfrak{X}^1(M), [[Z, [Z, \Pi]], \Pi] = 0 \quad \text{and} \quad [[Z, X_{\mathrm{H}}], \Pi] = 0\}.$$

It is the set of solutions of the Yang–Baxter–Poisson and Poisson–Hamilton equations and can be called the space of bi-Hamiltonian systems of deformation type associated with (M, Π, H).

Proposition 3.8. *If Z is a vector field which satisfies the two equations*

$$[[Z, [Z, \Pi]], \Pi] = 0 \quad \textit{and} \quad [[Z, X_{\mathrm{H}}], \Pi] = 0$$

then, for all infinitesimal automorphism Y *of* Π, *the field* $Z + Y$ *is again a solution of these two equations*.

Proof. For the first one, we have,

$$\begin{aligned}[][[Z + Y, [Z + Y, \Pi]], \Pi] &= [[Z + Y, [Z, \Pi]], \Pi] = [[Y, \Pi'], \Pi] \\ &= [Y, [\Pi', \Pi]] + [\Pi', [\Pi, Y]] = 0\end{aligned}$$

the two last equalities resulting, respectively, from the graded Jacobi identity, the compatibility of Π with the deformed tensor $\Pi' = [Z, \Pi]$ and the fact that Y is an infinitesimal automorphism of Π.

For the second one, it results from the classical fact that $[Y, X_{\mathrm{H}}]$ is a Hamiltonian vector field and so is a Π-automorphism, precisely $[Y, X_{\mathrm{H}}] = X_{Y \cdot H}$. $\square$

Obviously the relation $\sim$ defined on $\mathfrak{X}^1(M)$ by

$$\forall Z, \quad Z' \in \mathfrak{X}^1(M), \qquad Z \sim Z' \Leftrightarrow [Z - Z', \Pi] = 0$$

is an equivalence relation and because of the previous proposition, if $Z \in \mathcal{S}$, all field Z' equivalent to Z is again in $\mathcal{S}$ which it allows to consider the well-defined space $\mathfrak{M} = \mathcal{S}/\sim$ that we will call the *Yang–Baxter–Poisson moduli space*.

3.3. Poisson–Hamilton equation and cohomology

In previous section we have seen the classical Poisson cohomology, which is the good theoretical framework to study deformations of Poisson structures. Now, we want to introduce a new cohomology which is adapted to the bi-Hamiltonian systems of deformation type and more precisely to the condition (ii) of Proposition 3.6 This condition $[[Z, X_{\mathrm{H}}], \Pi] = 0$ has already been called the *Poisson–Hamilton equation*, shortly referred as (PHE).

Let for $k \in \mathbb{N}$ and (M, Π, H) a Hamiltonian system:

$$\delta^k_\Pi : \mathfrak{X}^k(M) \longrightarrow \mathfrak{X}^{k+1}(M), \qquad \Lambda \mapsto [\Pi, [X_{\mathrm{H}}, \Lambda]]. \tag{12}$$

Because of the graded Jacobi identity for the Schouten–Nijenhuis bracket and the X_{H}-invariance of Π, we can deduce the following easy lemma.

Lemma 3.9. *For all $k \in \mathbb{N}$, for all $\Lambda \in \mathfrak{X}^k(M)$, we have*

$$\delta^k_\Pi(\Lambda) = d^k_\Pi[X_{\mathrm{H}}, \Lambda] = [X_{\mathrm{H}}, d^k_\Pi \Lambda],$$

where d^k_Π indicates the cobord operator of the Poisson cohomology associated with Π.

Using this lemma, we obtain immediately that for all $k \in \mathbb{N}$,

$$\delta^{k+1}_\Pi \circ \delta^k_\Pi = 0$$

and so, we obtain a new cohomology associated with the cochains complex

$$\mathfrak{X}^0(M) \overset{\delta^0_\Pi}{\to} \mathfrak{X}^1(M) \overset{\delta^1_\Pi}{\to} \mathfrak{X}^2(M) \overset{\delta^2_\Pi}{\to} \mathfrak{X}^3(M) \longrightarrow \cdots$$

that we will note $H^*_{\Pi,H}(M)$ and call the *Poisson–Hamilton cohomology*. In terms of this cohomology, the space of solutions of the Poisson–Hamilton equation (PHE) is $\operatorname{Ker} \delta^1_\Pi$. But an element Y of $\operatorname{Im}\delta^0_\Pi$ satisfies for some function $f \in \mathcal{C}^\infty(M)$,

$$Y = [\Pi, [X_{\mathrm{H}}, f]] = X_{\{H,f\}},$$

thus is Hamiltonian, and so leads to a trivial deformation of Π. It results from these observations that the space of "interesting" solutions of (PHE) is $H^1_{\Pi,H}(M)$. Obviously this cohomology depends strongly on the function H. In the particular case where there is a non-Noether symmetry Z of X_{H}, the class of this symmetry defines a non-trivial element of $H^1_{\Pi,H}(M)$.

We can also remark, using the previous lemma, that the two cobord operators d_Π and δ_Π satisfy the following identities:

$$d_\Pi \circ \delta_\Pi = \delta_\Pi \circ d_\Pi = 0. \tag{13}$$

In particular,

$$\mathfrak{X}^0(M) \overset{d^0_\Pi,\delta^0_\Pi}{\rightarrow} \mathfrak{X}^1(M) \overset{d^1_\Pi,\delta^1_\Pi}{\rightarrow} \mathfrak{X}^2(M) \overset{d^2_\Pi,\delta^2_\Pi}{\rightarrow} \mathfrak{X}^3(M) \longrightarrow \cdots \tag{14}$$

defines a bicomplex [4,5], i.e. we have, with simplified notations:

$$d_\Pi \circ d_\Pi = 0, \qquad \delta_\Pi \circ \delta_\Pi = 0, \qquad d_\Pi\delta_\Pi + \delta_\Pi d_\Pi = 0.$$

It is important to note that, the cobord operator for Poisson–Hamilton cohomology is not a differential operator of order one. So, we are only in an algebraic situation of bicomplex and there is no Frölicher operator relying the two cobord operators of the bicomplex.

4. Deformations and action-angle coordinates

4.1. Arnold–Liouville systems

Let (M, Π, H) a Hamiltonian system where Π has a maximal rank and so is associated with a symplectic form ω. This system is said completely integrable in the sense of Arnold–Liouville – or an *Arnold–Liouville system* – if there is an involutive family of functionally independent constants of motion $f_1, \ldots, f_n$, where $n = \dim M/2$, so that the fibers of the application

$$F : M \longrightarrow \mathbb{R}^n, \quad x \mapsto (f_1(x), \ldots, f_n(x))$$

are compact sets. In this situation we know that, if T is a connected component of a fiber, then T is a n-dimensional torus; moreover, there is a tubular neighbourhood Ω of T in M and a Poisson isomorphism between Ω and $U \times \mathbb{T}^n$ where U is an open set of $\mathbb{R}^n$ and $U \times \mathbb{T}^n$ is endowed with the canonical Poisson structure $\Pi_{\mathrm{can}} = \sum_{i=1}^n \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial \theta_i}$, where we note $q_1, \ldots, q_n, \theta_1, \ldots, \theta_n$ coordinates on the product $U \times \mathbb{T}^n$. These coordinates are called *action-angle coordinates* and the functions H and $f_1, \ldots, f_n$ depend only on $q_1, \ldots, q_n$ (see for example [9]). These classical results show that the model of Arnold–Liouville integrable Hamiltonian system around a torus is $(U \times \mathbb{T}^n, \Pi_{\text{can}}, H(q))$; we say that the Hamiltonian function H is *non-degenerate* if its Hessian matrix has a maximal rank on U. This generic condition implies that, the only first integrals of X_{H} are basic functions for the fibration $U \times \mathbb{T}^n \longrightarrow U$ given by the first projection, i.e. are functions of $q_1, \ldots, q_n$. In the following, when we will say *Arnold–Liouville system*, we will refer to the system $(U \times \mathbb{T}^n, \Pi_{\text{can}}, H(q))$.

The Poisson cohomology of such a system (of its manifold) verifies

$$\begin{aligned} H^2_{\Pi}(U \times \mathbb{T}^n) &\simeq H^2_{\text{DR}}(U \times \mathbb{T}^n) \simeq (H^0_{\text{DR}}(\mathbb{T}^n) \otimes H^2_{\text{DR}}(U)) \oplus (H^1_{\text{DR}}(\mathbb{T}^n) \otimes H^1_{\text{DR}}(U)) \\ &\oplus (H^2_{\text{DR}}(\mathbb{T}^n) \otimes H^0_{\text{DR}}(U)) \simeq H^2_{\text{DR}}(\mathbb{T}^n) \end{aligned}$$

because of the isomorphism between the Poisson and De-Rham cohomologies in the symplectic case, and the triviality of the De-Rham cohomology of an open ball of $\mathbb{R}^n$. Now, because $(\mathrm{d}\theta_i \wedge \mathrm{d}\theta_j)_{i<j}$ is a basis for $H^2_{\text{DR}}(\mathbb{T}^n)$, we get by symplectic duality the basis $\left(\frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial q_j}\right)_{i<j}$ of $H^2_{\Pi}(U \times \mathbb{T}^n)$. So, if Π' is a 2-tensor compatible with Π, there are a field Z and some real constants c_{ij} so that

$$\Pi' = [Z, \Pi] + \sum_{i<j} c_{ij} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial q_j}.$$

4.2. *Deformation with a non-Noether symmetry*

In a general way, a symmetry of a dynamical system $\dot{x} = X(x)$ is a vector field Z satisfying $[Z, X] = 0$. In the Hamiltonian framework, if (M, Π) is a Poisson manifold and H a smooth function on M giving a Hamiltonian vector field X_{H}, we have already seen in previous section that Chavchanidze [3] defined a non-Noether symmetry as a field Z verifying conditions $[Z, X_{\text{H}}] = 0$ and $[Z, \Pi] \neq 0$. In this situation, he proved several results – in the case where Π has maximal rank – about the existence of an involutive family of first integrals for X_{H}, about the existence of a bi-Hamiltonian formulation and the existence of a Frölicher–Nijenhuis operator and so on. All these results are clear in the light of the previous section and classical results on bi-Hamiltonian systems [8].

As we have already seen, the infinitesimal deformation of a Poisson tensor Π by the means of a field Z, namely $\Pi' = [Z, \Pi]$, verifies $[\Pi, \Pi'] = 0$ and so, in the case where Π' is Poisson, we obtain a pair of compatible Poisson structures. Moreover, if Z is a symmetry, it is clear that Π' is X_{H}-invariant and the given Hamiltonian system has a bi-Hamiltonian formulation. Thus, in the case where Π has maximal rank, the $(1, 1)$-tensor field relying the two structures can be used to obtain an involutive family of constants of motion. In this case, perhaps we will obtain the complete integrability of the Hamiltonian system (we have already said in the introduction the double point of view about it: we can try to prove complete integrability with this method or more realistic, we can try to see if known Poisson-commuting first integrals can appear as eigenvalues of the associated recursion operator).

Now we want to prove that the notion of non-Noether symmetry is not relevant for a large class of Arnold–Liouville systems because by its means, the involutive family of first integrals mentioned above contains only constants functions.

In the context of action-angle coordinates, we have the following result [1,2].

Lemma 4.1. *For a non-degenerate Arnold–Liouville system, a field Z is a symmetry of X_{H}, i.e. $[Z, X_{\mathrm{H}}] = 0$ if, and only if, Z is written as*

$$Z = \sum_{i=1}^{n} a_i(q) \frac{\partial}{\partial \theta_i}.$$

We deduce the following proposition.

Proposition 4.2. *For a non-degenerate Arnold–Liouville system, the deformation of the Poisson structure by means of a (non-Noether) symmetry gives a bi-Hamiltonian structure whose recursion operator has a trivial spectrum.*

Proof. Indeed, it results from the previous lemma that the bivector $[Z, \Pi]$ has the form

$$[Z, \Pi] = \left[\sum_{i=1}^{n} a_i(q) \frac{\partial}{\partial \theta_i}, \sum_{j=1}^{n} \frac{\partial}{\partial q_j} \wedge \frac{\partial}{\partial \theta_j} \right] = - \sum_{i,j=1}^{n} \frac{\partial a_i}{\partial q_j} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial \theta_j}$$

and so, if $[Z, \Pi]$ is a Poisson structure, the $(1, 1)$-tensor field relying the two compatible Poisson structures has a trivial spectrum. □

Consequently, if a Hamiltonian system is completely integrable in the sense of Arnold–Liouville, with a non-degenerate Hamiltonian, it is impossible to prove its integrability by means of a bi-Hamiltonian formulation based upon a deformation of the initial Poisson structure by a non-Noether symmetry. Thus, deformations by non-Noether symmetries cannot bring any explanation to the Arnold–Liouville integrability (at least when H is non-degenerate).

4.3. Mastersymmetries and deformations

Recall that a mastersymmetry of a Hamiltonian system is a field Z satisfying $[[Z, X_{\mathrm{H}}], X_{\mathrm{H}}] = 0$ and $[Z, X_{\mathrm{H}}] \neq 0$.

In [14], Smirnov showed (by applying Lemma 4.1 to $[Z, X_{\mathrm{H}}]$) that, if $(M = U \times \mathbb{T}^n, \Pi, H)$ is non-degenerate, the condition $[[Z, X_{\mathrm{H}}], X_{\mathrm{H}}] = 0$ is satisfied if, and only if, the field Z is written as

$$Z = \sum_{i=1}^{n} a_i(q) \frac{\partial}{\partial \theta_i} + \sum_{i=1}^{n} b_i(q) \frac{\partial}{\partial q_i} \tag{15}$$

and so characterized mastersymmetries in this situation. Using this fact and some results obtained in [1,2] we can prove the following theorem.

Theorem 4.3. *Let* $(M = U \times \mathbb{T}^n, \Pi, H)$ *be a non-degenerate Arnold–Liouville system. If* Z *is a field and* $\Pi' = [Z, \Pi]$ *is a (compatible)* X_{H}*-invariant Poisson structure (and so provides a bi-Hamiltonian formulation to the system) then, modulo a Hamiltonian vector field,* Z *is written as*

$$Z = \sum_{i=1}^{n} a_i(q)\frac{\partial}{\partial\theta_i} + \sum_{i=1}^{n} b_i(q)\frac{\partial}{\partial q_i}.$$

In this case, Z must be a mastersymmetry if the spectrum of the associated recursion operator has a non-trivial spectrum.

Proof. Let

$$Z = \sum_{i=1}^{n} a_i(q,\theta)\frac{\partial}{\partial\theta_i} + \sum_{i=1}^{n} b_i(q,\theta)\frac{\partial}{\partial q_i}$$

be a general field. We want to prove that if $\Pi' = [Z, \Pi]$ is a X_{H}-invariant compatible Poisson structure then, modulo an infinitesimal automorphism of Π, the coefficients a_i and b_i depend only on q. A straightforward calculation gives

$$\Pi' = -\sum_{i<j}\left(\frac{\partial a_i}{\partial q_j} - \frac{\partial a_j}{\partial q_i}\right)\frac{\partial}{\partial\theta_i}\wedge\frac{\partial}{\partial\theta_j} - \sum_{i<j}\left(\frac{\partial b_j}{\partial\theta_i} - \frac{\partial b_i}{\partial\theta_j}\right)\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial q_j}$$
$$-\sum_{i,j}\left(\frac{\partial b_i}{\partial q_j} + \frac{\partial a_j}{\partial\theta_i}\right)\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial\theta_j}. \qquad (16)$$

For some (great) constant k, we are sure that $\Pi_k := k\Pi + \Pi'$ has a maximal rank and so defines a symplectic form ω_k. Moreover, we have also $[\Pi_k, \Pi] = 0$, so Π_k and Π are compatible. From a symplectic point of view, we have two compatible symplectic forms ω_{can} and ω_k, in the sense where the (1, 1)-tensor field J defined by the relation available for all vector fields X and Y, $\omega_k(X, Y) = \omega(JX, Y)$, has a vanishing Nijenhuis torsion and of course, ω_k is X_{H}-invariant. So, according to a study done in the context of symplectic forms [1,2], we know that, in this situation, the coefficients of ω_k are functions of only coordinates $q_1, \ldots, q_n$ and that the fibration $U \times \mathbb{T}^n \longrightarrow U$ is Lagrangian, i.e. ω_k has no term in $\mathrm{d}\theta_i \wedge \mathrm{d}\theta_j$. It results that Π_k and also Π' has coefficients which depend only on coordinates q and has no term in $\frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial q_j}$. So we have,

$$\forall\, i < j, \quad \frac{\partial a_i}{\partial q_j} - \frac{\partial a_j}{\partial q_i} = f_{ij}(q), \qquad (17)$$

$$\forall\, i < j, \quad \frac{\partial b_j}{\partial\theta_i} - \frac{\partial b_i}{\partial\theta_j} = 0, \qquad (18)$$

$$\forall\, i, j, \quad \frac{\partial b_i}{\partial q_j} + \frac{\partial a_j}{\partial\theta_i} = g_{ij}(q). \qquad (19)$$

The second of these relations means that for each fixed q, the 1-form $\alpha := \sum b_i \, \mathrm{d}\theta_i$ is closed and thus, according to the De-Rham cohomology of the torus, there is a function F and n functions $c_i(q)$ so that

$$\alpha = \partial_\theta F + \sum c_i(q) \, \mathrm{d}\theta_i,$$

where $\partial_\theta F$ is the differential of F as a function of θ (q is constant). Thus we have for all i,

$$b_i(q, \theta) = \frac{\partial F}{\partial \theta_i} + c_i(q). \tag{20}$$

Now consider the field $Z' = Z + X_F$ whose coefficients are, with obvious notations,

$$a'_i = a_i + \frac{\partial F}{\partial q_i}, \qquad b'_i = b_i - \frac{\partial F}{\partial \theta_i}.$$

We have,

$$\frac{\partial a'_i}{\partial \theta_j} = \frac{\partial a_i}{\partial \theta_j} + \frac{\partial^2 F}{\partial q_i \partial \theta_j} = g_{ji}(q) - \frac{\partial b_j}{\partial q_i} + \frac{\partial^2 F}{\partial q_i \partial \theta_j} = g_{ji}(q) - \frac{\partial c_j}{\partial q_i}(q).$$

Thus on each torus, the functions a'_i are affine functions, so are constants and finally $a'_i = a'_i(q)$. We have also,

$$\frac{\partial b'_i}{\partial \theta_j} = \frac{\partial b_i}{\partial \theta_j} - \frac{\partial^2 F}{\partial \theta_i \partial \theta_j} = 0$$

so $b'_i = b'_i(q)$. It results from these calculations that Z' has the announced form. Now, according to Proposition 4.2, if the spectrum of the recursion operator associated with this bi-Hamiltonian system is interesting, in particular non-trivial, Z' cannot be a symmetry but a mastersymmetry. □

Remark 4.4. The modification of a given field Z by a Hamiltonian does not change the deformed structure Π' and the associated recursion operator. So, the previous result joined to the initial remarks about the Poisson cohomology of $U \times \mathbb{T}^n$ show that, in the case of a non-degenerate Arnold–Liouville system, the only bi-Hamiltonian structures – with a non-trivial spectrum – are obtained by deformation of the initial Poisson tensor by means of a mastersymmetry. So, only mastersymmetries are interesting in the Arnold–Liouville context; this fact can be brought closer to that of Smirnov [14] who showed that for an Arnold–Liouville system the only generators Z of degree k, in the sense where

$$L^k_{X_\mathrm{H}} Z = 0 \quad \text{and} \quad L^{k-1}_{X_\mathrm{H}} Z \neq 0$$

are symmetries and mastersymmetries (i.e. $k = 1$ or 2). These two results show the essential role of mastersymmetries.

5. Some examples of deformation

5.1. The Toda lattice

The Toda lattice is a system of n particles on a line; if $q_1, \dots, q_n$ are positions of these particles and $p_1 = \dot{q}_1, \dots, p_n = \dot{q}_n$, it is a Hamiltonian system on the phase space $\mathbb{R}^{2n}$ with coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ endowed with the canonical Poisson structure

$$\Pi = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$$

and the Hamiltonian function H associated with the system is

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} \mathrm{e}^{2(q_i - q_{i+1})}.$$

This system appears as a bi-Hamiltonian system with the following second Poisson structure [6]:

$$\Pi' = \sum_{i=1}^{n-1} 2\,\mathrm{e}^{2(q_i - q_{i+1})} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i<j} \frac{\partial}{\partial q_j} \wedge \frac{\partial}{\partial q_i}.$$

We know from the previous sections that the bivector Π' is necessarily a deformation of Π, i.e. $\Pi' = [Z, \Pi]$ for a suitable field Z which is uniquely defined modulo a Hamiltonian field. We can take for Z the following:

$$Z = \sum_{i=1}^{n-1} \mathrm{e}^{2(q_i - q_{i+1})} \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} \frac{1}{2} p_i^2 \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i<j} p_i \frac{\partial}{\partial q_j}.$$

In order to obtain this field Z, it suffices to use symplectic duality provided by Π and work with differential forms: a suitable field Z appears in this new context as a 1-form α verifying $\mathrm{d}\alpha = \omega'$, where ω' is the 2-form corresponding to Π by means of this duality.

5.2. The relativistic Toda lattice

For the relativistic Toda lattice, we have again

$$\Pi = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i},$$

but the Hamiltonian function H associated with the system is now

$$H = \sum_{i=1}^{n} \mathrm{e}^{q_i - q_{i+1} + p_i} + \mathrm{e}^{p_i}.$$

Here a bi-Hamiltonian formulation is provided [13] by the Poisson bivector

$$\Pi' = \sum_{i=1}^{n-1} \mathrm{e}^{-p_i} \frac{\partial}{\partial q_i} \wedge \left(\frac{\partial}{\partial p_i} + \sum_{j=i+1}^{n} \frac{\partial}{\partial q_j} \right) + \Pi'',$$

where

$$\Pi'' = \sum_{i=1}^{n-1} \mathrm{e}^{(q_i - q_{i+1} - p_{i+1})} \left(\left(\frac{\partial}{\partial p_i} + \frac{\partial}{\partial q_{i+1}} \right) \wedge \left(\frac{\partial}{\partial p_{i+1}} + \sum_{j=i+2}^{n} \frac{\partial}{\partial q_j} \right) \right) - \mathrm{e}^{(q_i - q_{i+1} - p_{i+1})} \left(\frac{\partial}{\partial p_{i+1}} \wedge \sum_{j=i+2}^{n} \frac{\partial}{\partial q_j} \right).$$

We can give the following Z satisfying $\Pi' = [Z, \Pi]$,

$$Z = \sum_{i=1}^{n-1} \mathrm{e}^{-p_i} \left(\frac{\partial}{\partial p_i} - \sum_{j=i+1}^{n} \frac{\partial}{\partial q_j} \right) - \sum_{i=1}^{n-1} \mathrm{e}^{(q_i - q_{i+1} - p_{i+1})} \left(\frac{\partial}{\partial p_{i+1}} + \sum_{j=i+2}^{n} \frac{\partial}{\partial q_j} \right).$$

5.3. The Euler equation

Using a symmetric matrix A, we can obtain on the Lie algebra $\mathrm{so}(n)$, a new Lie bracket given by $[M, N]_A := MAN - NAM$. Such a bracket defines a Poisson structure on the dual space $\mathrm{so}(n)^*$ of $\mathrm{so}(n)$. By means of the symmetric bilinear form $\langle , \rangle$ defined by $\langle M, N \rangle := \mathrm{Tr}(MN)$, we can identify the Lie algebra $\mathrm{so}(n)$ with its dual space $\mathrm{so}(n)^*$. Then, we get a new Poisson bracket $\{\}_A$ on $\mathrm{so}(n)$. In the case where $A = I_n$, this Poisson structure is the Kirillov–Kostant–Souriau Poisson structure; let $\{,\}$ the corresponding bracket. In [12], the authors affirm that the Poisson bracket $-2\{,\}_A$ is obtained by deformation from $\{,\}$. Because for $n \geq 3$, $\mathrm{so}(n)$ is a semi-simple Lie algebra of compact type, this result is not surprising according to the end of Section 2.2. Moreover in [12], the authors give the field $Z(M) := AM + MA$ as a suitable field allowing this deformation. We now prove this fact.

Indeed, the flow of the field Z is given by $\phi_t(M) = \mathrm{e}^{tA} M \,\mathrm{e}^{tA}$. For $P \in \mathrm{so}(n)$, let us define on $\mathrm{so}(n)$ the real function f_P,

$$f_P(M) := \mathrm{Tr}(MP).$$

If P, Q and M belong to $\mathrm{so}(n)$, we have

$$\{f_P \circ \phi_t, f_Q \circ \phi_t\}(\phi_{-t}(M)) = -\mathrm{Tr}(\mathrm{e}^{-tA} M \,\mathrm{e}^{-tA} [\mathrm{e}^{tA} P \,\mathrm{e}^{tA}, \mathrm{e}^{tA} Q \,\mathrm{e}^{tA}]).$$

After some simplifications, we get,

$$\{f_P \circ \phi_t, f_Q \circ \phi_t\}(\phi_{-t}(M)) = -\mathrm{Tr}(MP \,\mathrm{e}^{2tA} Q - MQ \,\mathrm{e}^{2tA} P).$$

By differentiation at $t = 0$ of this last expression, we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \{f_P \circ \phi_t, f_Q \circ \phi_t\}(\phi_{-t}(M)) = -2\mathrm{Tr}(MPAQ - MQAP) = -2\{f_P, f_Q\}_A(M)$$

and so the announced result.

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